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A NEW FAMILY OF CIRCULAR MODELS: THE WRAPPED LAPLACE DISTRIBUTIONS

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Abstract

We introduce a new class of asymmetric circular distributions, obtained by wrapping skewed Laplace distributions on the real line, around a circle. We obtain explicit forms for the densities and distribution functions for this family, as well as their trigonometric moments and related parameters. We derive basic properties of these laws and illustrate their modeling potential using a classical data set on orientation of ants.

1. Introduction

We introduce a new class of non-symmetric circular distributions by wrapping an asymmetric Laplace distribution around the circumference of a unit circle. Recall that when a real random variable (r.v.) X with probability density function (p.d.f.) f and characteristic function (ch.f.) ϕ is wrapped, then the p.d.f. of the wrapped r.v.,

$$X_w = X(\text{mod } 2\pi) \quad (1)$$

has density of the form

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$$f_w(\theta) = \sum_{k=-\infty}^{\infty} f(\theta + 2k\pi), \quad \theta \in [0, 2\pi), \quad (2)$$

and the characteristic function (the discrete Fourier transform)

$$\phi_p = Ee^{ipX_w} = \phi(p), \quad p = 0, \pm 1, \pm 2, \dots \quad (3)$$

(see, e.g., Mardia and Jupp [15], Jammalamadaka and SenGupta [7]).

Circular distributions play an important role in modeling directional data which arise in various fields. See, e.g., the references cited above, among others. While the wrapped Cauchy, normal, and stable distributions have been studied extensively (see, e.g., Lévy [12], Gatto and Jammalamadaka [3]), relatively little has been done for the case of wrapped exponential or wrapped double-exponential (or Laplace) distributions, which produce equally interesting circular models. Perhaps a first step in this direction is taken in Jammalamadaka and Kozubowski [6], who introduced a wrapped exponential circular model and derived basic properties of this class of distributions. In this work, we introduce the wrapped Laplace distribution. The Laplace distribution and its various generalizations are becoming prominent, particularly for financial applications (see, e.g., Kotz et al. [9]). We believe that the circular analog of this family would find many interesting applications in directional data, since many circular data sets resemble the characteristic shape of the Laplace distribution, which is asymmetric around the sharp peak at the mode. As an example, we show in Figure 1 below the data on directions chosen by migrating birds (taken from Bruderer [2]). Noticing the obvious asymmetry around the mean, Batschelet [1] remarked that it may be caused by a mixture of two or more distributions (wrapped Laplace distribution can indeed be viewed as such a mixture). Similar data sets can be found for example in Schmidt-Koenig [18], Matthews [16], and Batschelet [1].

Following the wrapped exponential case studied in Jammalamadaka and Kozubowski [6] (from now on, we will refer to this as JK), which we briefly describe in Section 2, we develop a basic theory of wrapped Laplace distributions. We define this class in Section 3, and in Section 4

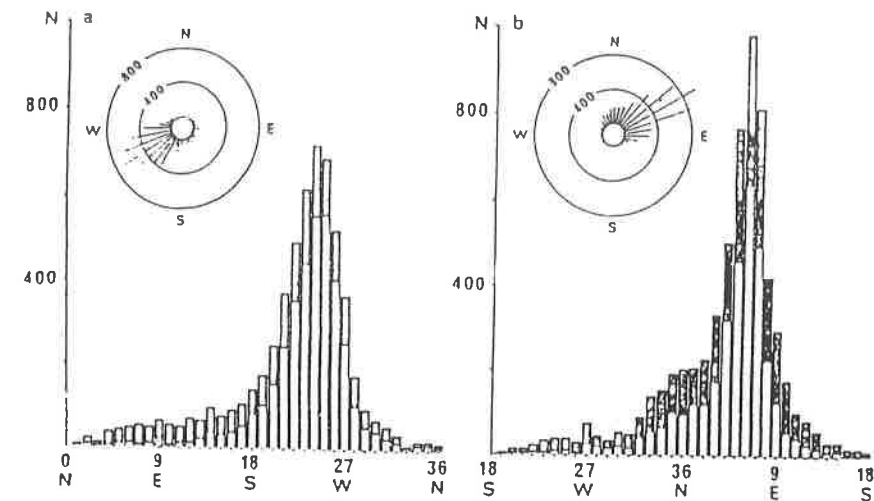


Figure 1. Linear and circular plots of directions chosen by migrating birds in autumn (a) and spring (b). White and shaded bars correspond to the frequencies during undisturbed nights and unfavorable conditions, respectively. Reproduced from Bruderer [2]

provide a characterization of the wrapped Laplace densities. Then, we derive their trigonometric moments and related parameters in Section 5, and discuss further properties of these laws in Section 6. In Sections 7 and 8, respectively, we briefly discuss parameter estimation and an application in biology illustrating the modeling potential of wrapped Laplace laws. Proofs are collected in Section 9.

2. A Wrapped Exponential Distribution

A wrapped exponential distribution is obtained by wrapping an exponential distribution with p.d.f.,

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0, \quad (4)$$

and ch.f.,

$$\phi(t) = \frac{1}{1 - it/\lambda}, \quad t \in R, \quad (5)$$

around a circumference of a unit circle.

Definition 2.1. A r.v. Θ on the unit circle is said to have a *wrapped exponential distribution* with parameter $\lambda > 0$, denoted by $WE(\lambda)$, if the ch.f. and the p.d.f. of Θ are given by

$$\phi_p = \frac{1}{1 - ip/\lambda}, \quad p = 0, \pm 1, \pm 2, \dots \quad (6)$$

and

$$f_w(\theta) = \lambda e^{-\lambda\theta} \sum_{k=0}^{\infty} [e^{-2\pi\lambda}]^k = \frac{\lambda e^{-\lambda\theta}}{1 - e^{-2\pi\lambda}}, \quad \theta \in [0, 2\pi), \quad (7)$$

respectively. We then write $\Theta \sim WE(\lambda)$.

The c.d.f. of the wrapped exponential distribution is

$$F_w(\theta) = \frac{1 - e^{-\lambda\theta}}{1 - e^{-2\pi\lambda}}, \quad \theta \in [0, 2\pi). \quad (8)$$

When $\lambda = 0$ we obtain the circular uniform distribution while for $\lambda < 0$ the distribution given by (7) results from wrapping the "negative" exponential distribution with parameter $|\lambda| > 0$, whose p.d.f. is

$$f(x) = |\lambda| e^{|\lambda|x}, \quad x < 0. \quad (9)$$

Thus, one can consider a more general class of distributions $WE(\lambda)$ with $\lambda \in R$, where we have the relation

$$\Theta \sim WE(\lambda) \text{ then } 2\pi - \Theta \sim WE(-\lambda). \quad (10)$$

Trigonometric moments and related parameters of these distributions admit explicit forms, see Table 1. As shown by JK, wrapped exponential distributions retain the important properties of infinite divisibility and maximum entropy of the corresponding exponential distribution. The entropy of a r.v. Θ with p.d.f. f is defined as

$$H(\Theta) = - \int_0^{2\pi} f(\theta) \ln f(\theta) d\theta, \quad (11)$$

and measures the uncertainty associated with the probability distribution

of Θ . Consider the class C of all circular r.v.'s with density f satisfying the condition

$$\int_0^{2\pi} \theta f(\theta) d\theta = m, \quad 0 < m < 2\pi. \quad (12)$$

Then, the maximum entropy is

$$\max_{\Theta \in C} H(\Theta) = \ln \left(\frac{1 - e^{-2\pi\lambda}}{\lambda} \right) + \frac{1}{\lambda} - \frac{2\pi e^{-2\pi\lambda}}{1 - e^{-2\pi\lambda}}, \quad (13)$$

and is attained by the $WE(\lambda)$ distribution with density (7), where $\lambda = (2\pi\xi)^{-1}$ and ξ satisfies the equation

$$\frac{m}{2\pi} = \xi - \frac{1}{e^{1/\xi} - 1}. \quad (14)$$

This is in contrast to the von Mises density which has the maximum entropy subject to the conditions

$$\int_0^{2\pi} (\cos \theta) f(\theta) d\theta = m_1, \quad \int_0^{2\pi} (\sin \theta) f(\theta) d\theta = m_2.$$

3. A Wrapped Laplace Distribution

Recall that the classical Laplace distribution is a symmetric distribution on R with p.d.f.,

$$f(x) = \frac{\lambda}{2} e^{-\lambda|x|}, \quad x \in R, \quad \lambda > 0. \quad (15)$$

Note that (15) can be written as a mixture

$$f(x) = pf_1(x) + (1-p)f_2(x), \quad x \in R, \quad (16)$$

where $p = 1/2$ and

$$f_1(x) = \lambda e^{-\lambda x} \quad (x > 0), \quad f_2(x) = \lambda e^{\lambda x} \quad (x < 0) \quad (17)$$

are the densities of exponential and negative exponential distributions, respectively. More generally, one can consider mixtures of the form (16)

with any $p \in (0, 1)$ and two distinct values λ_1 and λ_2 for the parameters of the exponential and the negative exponential distributions, leading to asymmetric Laplace laws (cf., McGill [17], Lingappaiah [13], Holla and Bhattacharya [5]). Then, the wrapped distribution corresponding to (16) takes the form

$$f_w(\theta) = pf_{w1}(\theta) + (1-p)f_{w2}(\theta), \quad \theta \in [0, 2\pi), \quad (18)$$

where f_{w1} and f_{w2} are the densities of the wrapped exponential laws $WE(\lambda_1)$ and $WE(-\lambda_2)$, corresponding to f_1 and f_2 (see (7)). Thus, the p.d.f. (18) takes an explicit form

$$f_w(\theta) = p \frac{\lambda_1 e^{-\lambda_1 \theta}}{1 - e^{-2\pi\lambda_1}} + (1-p) \frac{\lambda_2 e^{\lambda_2 \theta}}{e^{2\pi\lambda_2} - 1}, \quad \theta \in [0, 2\pi). \quad (19)$$

In particular, we shall focus on one specific class of asymmetric Laplace laws where in the representation (16) we put

$$p = \frac{1}{\kappa^2 + 1}, \quad \lambda_1 = \lambda\kappa, \quad \lambda_2 = \lambda/\kappa \quad (20)$$

for some $\kappa, \lambda > 0$. The asymmetric Laplace density (16) with the above parameters takes the form

$$f(x) = \lambda(1/\kappa + \kappa)^{-1} \begin{cases} e^{-\lambda\kappa|x|} & \text{for } x \geq 0, \\ e^{-(\lambda/\kappa)|x|} & \text{for } x < 0, \end{cases} \quad (21)$$

while the corresponding ch.f. is

$$\phi(t) = \frac{1}{1 + t^2/\lambda^2 - it(1/\kappa - \kappa)/\lambda} = \frac{1}{1 - it/(\lambda\kappa)} \frac{1}{1 + it/(\lambda/\kappa)}, \quad t \in R. \quad (22)$$

Note that for $\kappa = 1$ the above distribution reduces to the classical symmetric Laplace law with density (15). This class of asymmetric Laplace distributions, introduced in Hinkley and Revankar [4], has found numerous applications, particularly in mathematical finance (see, e.g., Madan et al. [14], Kozubowski and Podgórski [10, 11]). Kotz et al. [9] argue that the distribution with density (21) is the most natural non-symmetric generalization of the classical Laplace distribution and provide

an up to date theory and applications of these laws and their generalizations.

Let X have an asymmetric Laplace distribution with p.d.f. (21) and ch.f. (22). Then, by (19) and (20), the corresponding wrapped r.v. (1) has the p.d.f.,

$$f_w(\theta) = \frac{\lambda\kappa}{1 + \kappa^2} \left(\frac{e^{-\lambda\kappa\theta}}{1 - e^{-2\pi\lambda\kappa}} + \frac{e^{(\lambda/\kappa)\theta}}{e^{2\pi\lambda/\kappa} - 1} \right), \quad \theta \in [0, 2\pi), \quad (23)$$

and the ch.f.,

$$\phi_p = \frac{1}{1 + p^2/\lambda^2 - ip(1/\kappa - \kappa)/\lambda} = \frac{1}{1 - ip/(\lambda\kappa)} \frac{1}{1 + ip/(\lambda/\kappa)}, \quad (24)$$

$$p = 0, \pm 1, \pm 2, \dots$$

Formula (23) should be extended in a periodic fashion for the values of θ outside of the interval $[0, 2\pi)$.

Definition 3.1. A r.v. Θ on the unit circle is said to have a *wrapped Laplace distribution* with parameters $\lambda > 0$ and $\kappa > 0$, denoted by $WL(\lambda, \kappa)$, if the p.d.f. and the ch.f. of Θ are given by (23) and (24), respectively. We then write $\Theta \sim WL(\lambda, \kappa)$.

The corresponding distribution function (c.d.f.) is easily obtained by integrating (23) and is given by

$$F_w(\theta) = \frac{1}{1 + \kappa^2} \frac{1 - e^{-\lambda\kappa\theta}}{1 - e^{-2\pi\lambda\kappa}} + \frac{\kappa^2}{1 + \kappa^2} \frac{e^{(\lambda/\kappa)\theta} - 1}{e^{2\pi\lambda/\kappa} - 1}, \quad \theta \in [0, 2\pi). \quad (25)$$

Remark 3.1. Note that the p.d.f. (23) integrates to 1 on $[0, 2\pi)$ for any values of λ , including $\lambda < 0$, so that the definition can be extended to include all $\lambda \in R$. However, using the densities, it can be checked that

$$WL(-\lambda, \kappa) = WL(\lambda, 1/\kappa), \quad (26)$$

so that we may (and shall) restrict attention to the case $\lambda > 0$ in the sequel.

Remark 3.2. A more general class of circular distributions is obtained by wrapping the so called K -Bessel function distributions given by the ch.f. $[\phi(t)]^\tau$, $\tau > 0$, where ϕ is the asymmetric Laplace ch.f. (22). Here, the ch.f. of the corresponding wrapped distribution takes the form

$$\phi_p = \left(\frac{1}{1 - ip/(\lambda\kappa)} \right)^\tau \left(\frac{1}{1 + ip/(\lambda/\kappa)} \right)^\tau, \quad p = 0, \pm 1, \pm 2, \dots \quad (27)$$

For $\tau = 1$ the above distribution reduces to the wrapped Laplace distribution.

Remark 3.3. Note that by the construction, the wrapped Laplace $WL(\lambda, \kappa)$ r.v. Θ admits the following mixture representation:

$$\Theta \stackrel{d}{=} I\Theta_1 + (1 - I)\Theta_2, \quad (28)$$

where Θ_1 and Θ_2 are independent wrapped exponential $WE(\lambda\kappa)$ and $WE(-\lambda/\kappa)$ r.v.'s, and I is an indicator random variable (independent of Θ_1 and Θ_2) taking on the values 1 and 0 with probabilities $1/(1 + \kappa^2)$ and $\kappa^2/(1 + \kappa^2)$, respectively.

Remark 3.4. By the factorization of the asymmetric Laplace ch.f. (22), the corresponding r.v. has the same distribution as the difference of two independent exponential random variables (see, e.g., Kotz et al. [9]). Since the wrapped Laplace ch.f. (24) admits a similar factorization, we obtain an analogous representation for the wrapped Laplace r.v. $\Theta \sim WL(\lambda, \kappa)$:

$$\Theta \stackrel{d}{=} \Theta_1 + \Theta_2 \pmod{2\pi}, \quad (29)$$

where Θ_1 and Θ_2 are independent wrapped exponential $WE(\lambda\kappa)$ and $WE(-\lambda/\kappa)$ r.v.'s, as stated before.

Remark 3.5. The $WL(\lambda, \kappa)$ distribution converges weakly to the circular uniform distribution as $\lambda \rightarrow 0$ since the corresponding distribution function converges to $\theta/(2\pi)$. Also, as $\kappa \rightarrow 0^+$ or $\kappa \rightarrow \infty$, the

wrapped Laplace density (23) converges to $1/(2\pi)$, producing the limiting circular uniform distribution as well.

Remark 3.6. A three-parameter class of distributions can be defined by introducing a location parameter $\eta \in [0, 2\pi)$ and shifting the wrapped Laplace p.d.f. (defined on R by a periodic extension) by η with the resulting densities of the form

$$g(\theta) = f_w(\theta - \eta)$$

with f_w given by (23). Parameter η clearly corresponds to the mode (which differs from the mean) for such a family but for simplicity we shall restrict ourselves to the case $\eta = 0$ in the sequel.

The following result is a simple consequence of (10).

Lemma 3.1. *If $\Theta \sim WL(\lambda, \kappa)$, then $2\pi - \Theta \sim WE(\lambda, 1/\kappa)$.*

In case of a symmetric Laplace distribution with $\kappa = 1$, the wrapped Laplace density (23) simplifies to

$$\frac{\lambda}{2} \frac{e^{(2\pi-\theta)\lambda} + e^{\lambda\theta}}{e^{2\pi\lambda} - 1}, \quad (30)$$

and we have $2\pi - \Theta \stackrel{d}{=} \Theta$ for $\Theta \sim WL(\lambda, 1)$.

4. Characterization of the Densities

Let $f(\cdot; \lambda, \kappa)$ denote the density (23) of the $WL(\lambda, \kappa)$ distribution. The basic properties of $f(\cdot; \lambda, \kappa)$ are described in the following lemma, which is proved in Section 9.

Lemma 4.1. *Let $f(\cdot; \lambda, \kappa)$ be the density (23) of the $WL(\lambda, \kappa)$ distribution with $\lambda > 0$. Then*

(i) $f(\cdot; \lambda, \kappa)$ is strictly decreasing on $(0, \theta^*)$ and strictly increasing on $(\theta^*, 2\pi)$, where

$$\theta^* = \left(\frac{\lambda}{\kappa} + \lambda\kappa \right)^{-1} \ln \left(\kappa^2 \frac{e^{2\pi\lambda/\kappa} - 1}{1 - e^{-2\pi\lambda\kappa}} \right). \quad (31)$$

Moreover,

$$\theta^* > \pi \text{ for } \kappa < 1, \theta^* = \pi \text{ for } \kappa = 1, \text{ and } \theta^* < \pi \text{ for } \kappa > 1; \quad (32)$$

(ii) The maximum and the minimum values of $f(\cdot; \lambda, \kappa)$ are

$$f(0; \lambda, \kappa) = \lim_{\theta \rightarrow 2\pi} f(\theta; \lambda, \kappa) = \frac{\lambda\kappa}{1 + \kappa^2} \left(\frac{1}{e^{2\pi\lambda/\kappa} - 1} + \frac{1}{1 - e^{-2\pi\lambda\kappa}} \right) \quad (33)$$

and

$$f(\theta^*; \lambda, \kappa) = e^{1+\kappa^2} \left(\frac{e^{2\pi\lambda/\kappa} - 1}{2\pi\lambda/\kappa} \right)^{-\kappa^2} \left(\frac{e^{2\pi\lambda\kappa} - 1}{2\pi\lambda\kappa} \right)^{-1}, \quad (34)$$

respectively.

(iii) For any given $\lambda \geq 0$ and $\kappa \geq 0$, we have

$$f(\theta; \lambda, 1/\kappa) = f(2\pi - \theta; \lambda, \kappa), \quad \theta \in [0, 2\pi). \quad (35)$$

Remark 4.1. It appears from the graphs of θ^* as a function of κ (done in MAPLE), that for a given $\lambda > 0$, as κ increases from 0 to ∞ then the value of θ^* given by (31) decreases monotonically from 2π to 0. Similar graphs of $f(0; \lambda, \kappa)$ as a function of κ show that the values at the mode given by (33) are increasing monotonically from the lowest value of

$$f(0; \lambda, 0) = \frac{1}{2\pi}$$

to the highest value of

$$f(0; \lambda, 1) = \frac{\lambda e^{2\pi\lambda} + 1}{2 e^{2\pi\lambda} - 1}$$

as the parameter κ is increasing from 0 to 1 (and they decrease monotonically between the same values as κ is increasing from 1 to ∞). At present we do not have formal proofs of these results.

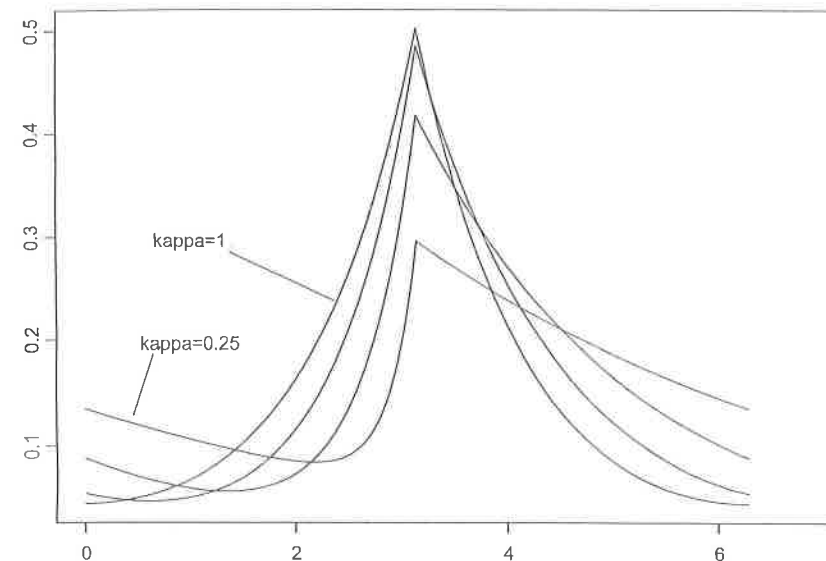


Figure 2. Wrapped Laplace densities with the mode at $\eta = \pi$, $\lambda = 1$, and κ equal to 0.25, 0.5, 0.75, 1

In Figure 2 above we present densities of the $WL(\lambda, \kappa)$ distributions with $\lambda = 1$ and selected values of κ . We can clearly observe the characteristic shape of the densities with their sharp peak at the mode. We also see how the minimum point θ^* and the minimum value $f(\theta^*; \lambda, \kappa)$ both decrease as κ increases from $0 \rightarrow 1$ while at the same time the value of the density at the mode increases.

5. Trigonometric Moments and Related Parameters

Computation of the trigonometric moments α_p and β_p , where

$$\phi_p = \alpha_p + i\beta_p, \quad p = 0, \pm 1, \pm 2, \dots, \quad (36)$$

and related parameters of the wrapped Laplace distribution $WL(\lambda, \kappa)$ is straightforward. For example, by utilizing the mixture representation (28) and formulas for the trigonometric moments of the wrapped exponential distributions $WE(\lambda\kappa)$ and $WE(-\lambda/\kappa)$ (see Table 1), we obtain

$$\begin{aligned}\alpha_p &= \frac{1}{1+\kappa^2} \frac{\kappa^2 \lambda^2}{\kappa^2 \lambda^2 + p^2} + \frac{\kappa^2}{1+\kappa^2} \frac{(-\lambda/\kappa)^2}{(-\lambda/\kappa)^2 + p^2} \\ &= \frac{\kappa^2 \lambda^2 (p^2 + \lambda^2)}{(\lambda^2 \kappa^2 + p^2)(\kappa^2 p^2 + \lambda^2)}, \quad p = 0, \pm 1, \dots,\end{aligned}\quad (37)$$

$$\begin{aligned}\beta_p &= \frac{1}{1+\kappa^2} \frac{p\kappa\lambda}{\kappa^2 \lambda^2 + p^2} + \frac{\kappa^2}{1+\kappa^2} \frac{(-\lambda/\kappa)p}{(-\lambda/\kappa)^2 + p^2} \\ &= \frac{p\kappa\lambda^3(1-\kappa^2)}{(\lambda^2 \kappa^2 + p^2)(\kappa^2 p^2 + \lambda^2)}, \quad p = 0, \pm 1, \dots\end{aligned}\quad (38)$$

Note that since the ϕ_p 's are the Fourier coefficients, one can write the density $f_w(\theta)$ in the form

$$f_w(\theta) = \sum_p \phi_p e^{-ip\theta}, \quad \theta \in [0, 2\pi). \quad (39)$$

Thus, the density of the $WL(\lambda, \kappa)$ distribution admits the Fourier representation

$$f(\theta; \lambda, \kappa) = \frac{1}{2\pi} \left[1 + 2 \sum_{p=1}^{\infty} \frac{\kappa\lambda[\kappa\lambda(p^2 + \lambda^2) \cos p\theta + p\lambda^2(1 - \kappa^2) \sin p\theta]}{(\lambda^2 \kappa^2 + p^2)(\kappa^2 p^2 + \lambda^2)} \right]. \quad (40)$$

Similarly, to compute the parameters ρ_p and μ_p^0 of the polar representation

$$\phi_p = \rho_p e^{i\mu_p^0}, \quad p \geq 0, \quad (41)$$

for the wrapped Laplace distribution we can exploit the factorization (24) of the $WL(\lambda, \kappa)$ ch.f. combining it with the formulas for the corresponding parameters of the $WE(\lambda)$ distribution (see Table 1). In particular, the resultant length is

$$\rho = \rho_1 = \frac{\lambda^2}{\sqrt{1 + \lambda^2 \kappa^2} \sqrt{\lambda^2/\kappa^2 + 1}}, \quad (42)$$

and the mean direction is

$$\mu_0 = \mu_1^0 = \begin{cases} \tan^{-1}(1/(\lambda\kappa)) - \tan^{-1}(\kappa/\lambda) & \text{for } \kappa \leq 1, \\ 2\pi + \tan^{-1}(1/(\lambda\kappa)) - \tan^{-1}(\kappa/\lambda) & \text{for } \kappa > 1. \end{cases} \quad (43)$$

Note that the mean direction lies in the interval $[0, \pi/2)$ for $\kappa \leq 1$ and in the interval $[3\pi/2, 2\pi)$ for $\kappa > 1$. This restriction on the mean is caused by the fact that the mode is equal to zero.

A population median direction ξ_0 of a circular distribution with density f is any solution (in the interval $[0, 2\pi)$) of

$$\int_{\xi_0}^{\xi_0 + \pi} f(\theta) d\theta = \int_{\xi_0 + \pi}^{\xi_0 + 2\pi} f(\theta) d\theta = \frac{1}{2}, \quad (44)$$

where the density f satisfies

$$f(\xi_0) > f(\xi_0 + \pi), \quad (45)$$

see, e.g., Mardia and Jupp [15]. It is clear that the median direction of the wrapped symmetric Laplace distribution $WL(\lambda, 1)$ is equal to zero (and coincides with the mean direction and the mode). In the following result, proved in Section 9, we describe the median direction for the general case.

Proposition 5.1. *Let $\Theta \sim WL(\lambda, \kappa)$. Then, Θ admits a unique median direction given by*

$$\xi_0 = \begin{cases} \xi^* & \text{for } \lambda > 0, 0 < \kappa < 1, \\ \xi^* + \pi & \text{for } \lambda > 0, \kappa > 1, \end{cases} \quad (46)$$

where $\xi^* \in [0, \pi]$ is the unique solution of the equation

$$\frac{1}{1 + \kappa^2} \left(\frac{e^{-\lambda\kappa\xi}}{1 + e^{-\lambda\kappa\pi}} + \kappa^2 \frac{e^{\lambda\xi/\kappa}}{1 + e^{\lambda\pi/\kappa}} \right) = \frac{1}{2}. \quad (47)$$

Clearly, the median is less than π for $\kappa < 1$ and greater than π for $\kappa > 1$.

Other parameters, including circular variance and standard deviation, central trigonometric moments, and circular skewness and

kurtosis, can be computed with ease. The values are summarized in Table 2.

6. Divisibility Properties

In this section, we show that wrapped Laplace distributions share the well known divisibility properties of the Laplace laws on the real line.

6.1. Infinite divisibility

Recall that an angular r.v. Θ (and its probability distribution) is said to be *infinitely divisible* if for any integer $n \geq 1$ there exist i.i.d. angular r.v.'s $\Theta_1, \dots, \Theta_n$ such that

$$\Theta_1 + \dots + \Theta_n \pmod{2\pi} \stackrel{d}{=} \Theta. \quad (48)$$

As remarked by Mardia and Jupp [15], a circular variable obtained by wrapping an infinitely divisible random variable is infinitely divisible. Thus, infinite divisibility of the wrapped Laplace distribution follows immediately from that of the Laplace distribution.

Proposition 6.1. *If $\Theta \sim WL(\lambda, \kappa)$, where $\lambda, \kappa \geq 0$, then Θ is infinitely divisible. Moreover, for any positive integer $n \geq 1$ the equality in distribution (48) holds with uniform circular variable Θ_1 if either $\lambda = 0$ or $\kappa = 0$, and otherwise with*

$$\Theta_1 \stackrel{d}{=} \Theta' + \Theta'' \pmod{2\pi}, \quad (49)$$

where Θ' and Θ'' are independent wrapped gamma r.v.'s with ch.f.'s

$$\left(\frac{1}{1 - ip/(\lambda\kappa)} \right)^{1/n} \text{ and } \left(\frac{1}{1 + ip/(\lambda/\kappa)} \right)^{1/n}, \quad (50)$$

respectively.

6.2. Geometric infinite divisibility

Recall the notion of geometric infinite divisibility for angular distributions, introduced in JK.

Definition 6.1. An angular r.v. Θ is said to be *geometric infinitely divisible* if for any $q \in (0, 1)$ there exist i.i.d. angular r.v.'s $\Theta_1, \Theta_2, \dots$ such that

$$\Theta_1 + \dots + \Theta_{v_q} \pmod{2\pi} \stackrel{d}{=} \Theta, \quad (51)$$

where v_q has the geometric distribution

$$P(v_q = k) = (1 - q)^{k-1}q, \quad k = 1, 2, 3, \dots \quad (52)$$

It is easy to see that geometric infinite divisibility of a real r.v. X implies the same property for the wrapped r.v. X_w . Since Laplace distributions on the real line are geometric infinitely divisible (see, e.g., Kotz et al. [9]), the same property is shared by the class of wrapped Laplace distributions.

Proposition 6.2. *If $\Theta \sim WL(\lambda, \kappa)$, where $\lambda, \kappa \geq 0$, then Θ is geometric infinitely divisible. Moreover, for any $q \in (0, 1)$ the equality in distribution (51) holds where the Θ_i 's have the uniform circular distribution for $\lambda = 0$ or $\kappa = 0$, and the $WL(\lambda_q, \kappa_q)$ distribution for $\lambda, \kappa > 0$, where*

$$\lambda_q = \lambda/\sqrt{q} \quad (53)$$

and κ_q is the unique solution of the equation

$$\frac{1}{\kappa_q} - \kappa_q = \sqrt{q} \left(\frac{1}{\kappa} - \kappa \right). \quad (54)$$

7. Estimation

Although wrapped Laplace density is available in closed form, maximum likelihood estimators must be found via a numerical search. Alternatively, one can obtain moment estimators of wrapped Laplace parameters in closed form. Below we briefly outline the method of

moments, assuming for simplicity that the mode is at zero (otherwise the data need to be shifted by the sample mode). Let $\theta_1, \dots, \theta_n$ be a random sample from $WL(\lambda, \kappa)$ distribution. In the symmetric case ($\kappa = 1$), equate the theoretical resultant length (42) with its sample counterpart

$$r = \sqrt{\left(\frac{1}{n} \sum_{j=1}^n \cos \theta_j\right)^2 + \left(\frac{1}{n} \sum_{j=1}^n \sin \theta_j\right)^2} \quad (55)$$

to obtain

$$\hat{\lambda} = \sqrt{\frac{r}{1-r}} \quad (56)$$

In the general case, we can proceed similarly by equating the theoretical quantities ρ_1 and ρ_2 (see Table 2) with the corresponding empirical values r_1 given by the right-hand-side of (55) and

$$r_2 = \sqrt{\left(\frac{1}{n} \sum_{j=1}^n \cos(2\theta_j)\right)^2 + \left(\frac{1}{n} \sum_{j=1}^n \sin(2\theta_j)\right)^2}$$

Using the values from Table 2, we obtain the system of equations:

$$\begin{cases} \lambda^4 = (\lambda^2 \kappa^2 + 1)(\lambda^2 \kappa^{-2} + 1)r_1^2, \\ \lambda^4 = (\lambda^2 \kappa^2 + 4)(\lambda^2 \kappa^{-2} + 4)r_2^2. \end{cases} \quad (57)$$

Multiplying the first and the second equations by $4r_2^2$ and r_1^2 , respectively, and subtracting the corresponding sides we obtain a simple equation for λ that yields the solution

$$\hat{\lambda} = \left(\frac{12r_1^2 - r_2^2}{3r_1^2 r_2^2 - 4r_2^2 + r_1^2}\right)^{1/4} \quad (58)$$

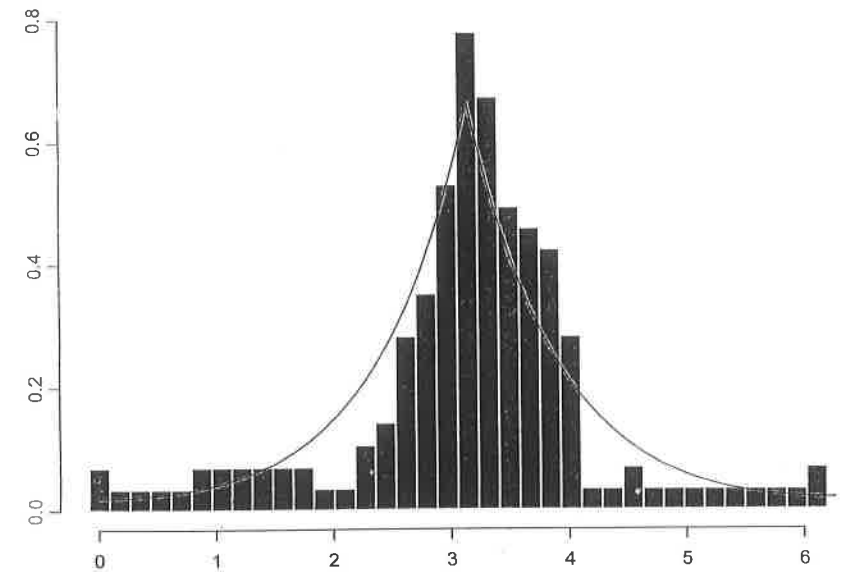


Figure 3. Data on optical orientation of ants in an arena (taken from Jander [8]) along with the fitted wrapped Laplace distribution with mode at π and $\kappa = 1, \lambda = 1.3$

Substituting this value into the first equation in (57) results in the following equation quadratic in κ :

$$\kappa^4(\hat{\lambda}^2 r_1^2) + \kappa^2(r_1^2 + r_1^2 \hat{\lambda}^4 - \hat{\lambda}^4) + \hat{\lambda}^2 r_1^2 = 0, \quad (59)$$

which yields the solution

$$\hat{\kappa} = \sqrt{\frac{\hat{\lambda}^4 - r_1^2(1 + \hat{\lambda}^4) + \sqrt{(r_1^2 + r_1^2 \hat{\lambda}^4 - \hat{\lambda}^4)^2 - 4\hat{\lambda}^4 r_1^4}}{2\hat{\lambda}^2 r_1^2}} \quad (60)$$

8. An Application in Biology

Our family of distributions may be useful in modeling skew and/or peaked circular data. Such data frequently results from orientation experiments in biology (see, e.g., Jander [8], Matthews [16], Schmidt-Koenig [18], Bruderer [2], Batschelet [1]). To illustrate the modeling potential of wrapped Laplace laws, we consider the ant orientation data

reported in Jander [8] and discussed in Batschelet [1]. The ants (*Formica rufa* L.) were put in an arena where they could see a black target, and they run towards the target. We recovered the data from Figure 3.2.2 in Batschelet, obtaining the mean vector length of $r = 0.6374$. Assuming a symmetric wrapped Laplace model (with $\kappa = 1$), we used the moment estimator (56) of λ obtaining $\hat{\lambda} = 1.3$. A (linear) histogram of the data along with its fitted symmetric wrapped Laplace density appears in Figure 3. It appears that the wrapped Laplace distribution provides a better model for these data than the von Mises distribution fitted by Batschelet [1].

9. Proofs

To prove Lemma 4.1 we need a few auxiliary results.

Lemma 9.1. *The function*

$$g(x) = \begin{cases} \frac{e^x - 1}{x} & \text{for } x > 0, \\ 1 & \text{for } x = 0, \end{cases} \quad (61)$$

is continuous and monotonically increasing on $[0, \infty)$ with the range of $[1, \infty)$ and

$$1 \leq g(x) \leq e^x, \quad x \geq 0. \quad (62)$$

Proof. The continuity and the range of g are obvious. Since

$$g'(x) = \frac{e^x(x-1)+1}{x^2} > 0, \quad x > 0, \quad (63)$$

the function g is strictly increasing. The left inequality in (62) follows from the definition and the monotonicity of g . To establish the right inequality, we proceed by showing

$$h_1(x) = \frac{e^x - 1}{e^x} \leq x = h_2(x), \quad x > 0. \quad (64)$$

The latter inequality is obvious since the two functions h_1 and h_2 are

differentiable on $[0, \infty)$, $h_1(0) = h_2(0)$, and $h_1'(x) = e^{-x} \leq 1 = h_2'(x)$ for $x > 0$. The proof of the lemma is complete.

Lemma 9.2. *Let*

$$h(x, y) = x \frac{e^y - 1}{1 - e^{-xy}}, \quad x, y > 0. \quad (65)$$

Then,

$$1 \leq h(x, y) \leq e^{(x+1)y}, \quad x, y > 0. \quad (66)$$

Proof. Note that

$$h(x, y) = e^{xy} \frac{g(y)}{g(xy)}, \quad (67)$$

where g is the function defined by (61). To establish the left inequality in (66), observe that $e^{xy}/g(xy) \geq 1$ by Lemma 9.1, so that by (67) we have $h(x, y) \geq g(y)$. Since $g(y) \geq 1$ again by Lemma 9.1, the left inequality in (66) follows. The proof of the right inequality is similar. By Lemma 9.1, we have $g(xy) \geq 1$ and $g(y) \leq e^y$, so that $g(y)/g(xy) \leq e^y$. Now, the right inequality in (66) follows from the representation (67).

Proof of Lemma 4.1. We start with part (i). The first derivative of $f(\theta; \lambda, \kappa)$ is

$$\frac{d}{d\theta} f(\theta; \lambda, \kappa) = \frac{\kappa^2 \lambda^2}{1 + \kappa^2} \left(-\frac{e^{-\lambda\kappa\theta}}{1 - e^{-2\pi\lambda\kappa}} + \frac{1}{\kappa^2} \frac{e^{\lambda\theta/\kappa}}{e^{2\pi\lambda/\kappa} - 1} \right). \quad (68)$$

Thus, the derivative is positive whenever

$$e^{\lambda(\kappa+1/\kappa)\theta} > \kappa^2 \frac{e^{2\pi\lambda/\kappa} - 1}{1 - e^{-2\pi\lambda\kappa}}. \quad (69)$$

Note that by Lemma 9.2 the right-hand-side of (69) is greater than or equal to 1. Indeed, denoting $x = \kappa^2$ and $y = 2\pi\lambda/\kappa$, the right-hand-side of (69) has the form $h(x, y)$, where h is the function defined in Lemma

9.2. Thus, the solution of (69) is given by $\theta > \theta^*$, where θ^* is given by (31). It remains to show that θ^* is actually contained in the interval $[0, 2\pi]$. But this is evident from the representation

$$\theta^* = \frac{2\pi}{y(1+x)} \ln h(x, y), \quad (70)$$

where h , x and y are as before, when we invoke the relation (66) of Lemma 9.2.

We now move to (32). For $\kappa = 1$, we obtain $\theta^* = \pi$ by direct substitution. Next, consider $0 < \kappa < 1$. By (70) and the fact that $h(x, y) \geq g(y)$, where g is defined in (61), we conclude that

$$\theta^* \geq \frac{2\pi \ln g(y)}{y(1+x)}, \quad (71)$$

where h , x and y are as before. Consequently, since $0 < x = \kappa^2 < 1$, the inequality

$$\theta^* \geq \frac{\pi \ln g(y)}{y} \quad (72)$$

holds for all $y > 0$. It is easy to verify (by a repeated application of l'Hospital's rule) that as $y \rightarrow \infty$ then the ratio $\ln g(y)/y$ converges to 1, and consequently by (72) we must have $\theta^* \geq \pi$ as claimed. The proof of the case $\kappa > 1$ is similar. This completes the proof of (i).

We now establish to Part (ii). Substituting $\theta = 0$ into the p.d.f. (23) we obtain the right-hand-side of (33), or equivalently,

$$\frac{\lambda\kappa}{1+\kappa^2} \frac{e^{2\pi\lambda\kappa} - e^{-2\pi\lambda/\kappa}}{e^{2\pi\lambda\kappa} + e^{-2\pi\lambda/\kappa} - e^{2\pi\lambda(\kappa-1/\kappa)} - 1}, \quad (73)$$

for the value $f(0; \lambda, \kappa)$. The limit of $f(\theta; \lambda, \kappa)$ as $\theta \rightarrow 2\pi$ is equal to

$$\frac{\lambda\kappa}{1+\kappa^2} \left(\frac{1}{e^{2\pi\lambda\kappa} - 1} + \frac{1}{1 - e^{-2\pi\lambda/\kappa}} \right), \quad (74)$$

or equivalently,

$$\frac{\lambda\kappa}{1+\kappa^2} \frac{e^{2\pi\lambda/\kappa} - e^{-2\pi\lambda\kappa}}{e^{2\pi\lambda/\kappa} + e^{-2\pi\lambda\kappa} - e^{-2\pi\lambda(\kappa-1/\kappa)} - 1}. \quad (75)$$

Multiplying the numerator and the denominator of (75) by the factor $e^{2\pi\lambda(\kappa-1/\kappa)}$, we obtain (73), showing the equality in (33). The equality (34) is obtained by substituting θ^* given by (31) into the p.d.f. (23) and simplifying.

Finally, the proof of part (iii) is straightforward.

Proof of Proposition 5.1. For $\xi \in [0, \pi]$ define a function

$$g(\xi) = \int_{\xi}^{\xi+\pi} f_w(\theta) d\theta, \quad (76)$$

where f_w is the density (23) of the $WL(\lambda, \kappa)$ distribution. Clearly, if $g(0) \leq 1/2$, then $g(\pi) \geq 1/2$, and if $g(0) \geq 1/2$, then $g(\pi) \leq 1/2$, so there exists at least one $\xi^* \in [0, \pi]$ such that $g(\xi^*) = 1/2$. We shall show that there is exactly one such ξ^* and such that

$$f_w(\xi^*) > f_w(\xi^* + \pi) \text{ for } 0 < \kappa < 1 \text{ and } \lambda > 0 \quad (77)$$

and

$$f_w(\xi^*) < f_w(\xi^* + \pi) \text{ for } \kappa > 1 \text{ and } \lambda > 0. \quad (78)$$

Note that

$$g(\xi) = \frac{1}{1+\kappa^2} \left(\frac{e^{-\lambda\kappa\xi}}{1+e^{-\lambda\kappa\pi}} + \kappa^2 \frac{e^{\lambda\xi/\kappa}}{1+e^{\lambda\pi/\kappa}} \right), \quad (79)$$

so that $g(\xi) = 1/2$ is equivalent to (47), and

$$\frac{d}{d\xi} g(\xi) = f_w(\xi + \pi) - f_w(\xi) = \frac{\lambda\kappa}{1+\kappa^2} \left(\frac{e^{\lambda\xi/\kappa}}{1+e^{\lambda\pi/\kappa}} - \frac{e^{-\lambda\kappa\xi}}{1+e^{-\lambda\kappa\pi}} \right). \quad (80)$$

We shall establish the following two claims:

Claim 1. The function g is decreasing on $(0, \xi_m)$ and increasing on (ξ_m, π) , where

$$0 < \xi_m = \frac{1}{\lambda(1/\kappa + \kappa)} \ln \left(\frac{1 + e^{\lambda\pi/\kappa}}{1 + e^{-\lambda\pi\kappa}} \right) < \pi. \quad (81)$$

Claim 2. We have

$$g(0) > 1/2 \text{ for } 0 < \kappa < 1 \text{ and } \lambda > 0 \quad (82)$$

and

$$g(0) < 1/2 \text{ for } \kappa > 1 \text{ and } \lambda > 0. \quad (83)$$

Claims 1 and 2 produce the result. Indeed, if $0 < \kappa < 1$ and $\lambda > 0$, then the solution $\xi = \xi^*$ of $g(\xi) = 1/2$ satisfies $0 < \xi^* < \xi_m < \pi$. But then, since g decreases on $(0, \xi_m)$, we have $g'(\xi^*) = f_w(\xi^* + \pi) - f_w(\xi^*) < 0$, so that the relation (77) holds, implying that $\xi_0 = \xi^*$ is the median. Similarly, if $\kappa > 1$ and $\lambda > 0$, then since $g(0) < 1/2$, the solution $\xi = \xi^*$ of $g(\xi) = 1/2$ satisfies $0 < \xi_m < \xi^* < \pi$. Here, the function g increases on (ξ_m, π) , so that $g'(\xi^*) = f_w(\xi^* + \pi) - f_w(\xi^*) > 0$, producing (78) and consequently implying that the median is equal to $\xi^* + \pi$. It remains to establish the two claims.

We start with Claim 1. The derivative (80) is positive whenever

$$e^{\lambda(1/\kappa + \kappa)\xi} > \frac{1 + e^{\lambda\pi/\kappa}}{1 + e^{-\lambda\pi\kappa}}. \quad (84)$$

Let ξ_m be the infimum of the set of all ξ 's for which the relation (84) holds. Then, we have $0 < \xi_m < \pi$, since the right-hand-side of (84) is greater than 1, the function of ξ defined by the left-hand-side of (84) is increasing, and the relation (84) holds for $\xi = \pi$ (as can be verified easily). Claim 1 follows.

We shall now establish Claim 2. Consider $g(0)$ as a function of λ with

κ held fixed,

$$u(\lambda) = g(0) = \frac{1}{1 + \kappa^2} \left(\frac{1}{1 + e^{-\lambda\kappa\pi}} + \kappa^2 \frac{1}{1 + e^{\lambda\pi/\kappa}} \right). \quad (85)$$

Then, $u(0) = 1/2$ and

$$\frac{du(\lambda)}{d\lambda} = \frac{\pi\kappa(e^{\lambda\pi/\kappa} - e^{-\lambda\pi\kappa})(e^{\lambda\pi(1/\kappa - \kappa)} - 1)}{(1 + \kappa^2)(1 + e^{-\lambda\pi\kappa})^2(1 + e^{\lambda\pi/\kappa})^2}. \quad (86)$$

Now, for $0 < \kappa < 1$, we have $1/\kappa - \kappa > 0$ and thus $u'(\lambda) > 0$, so that $u(\lambda) > 1/2$ for $\lambda > 0$. Similarly, for $\kappa > 1$, we have $1/\kappa - \kappa < 0$ and thus $u'(\lambda) < 0$, producing $u(\lambda) < 1/2$ for $\lambda > 0$. This concludes the proof of Claim 2, and the result follows.

Proof of Proposition 6.2. The proof is similar to that for exponential distribution (see JK). First, assume that $\lambda = 0$ or $\kappa = 0$, in which case Θ has the circular uniform distribution. Then, the equality (51) holds where Θ_i 's have the circular uniform distribution as well. Consider now $\lambda, \kappa > 0$, and let Θ_i 's be i.i.d. variables with the $WE(\lambda_q, \kappa_q)$ distribution with the parameters specified by (53) and (54). Let $\tilde{\phi}_p$ be the ch.f. of Θ_1 . Then, conditioning on the distribution of v_q , we can write the ch.f. of the left-hand-side in (51) as follows:

$$\begin{aligned} Ee^{ip(\Theta_1 + \dots + \Theta_{v_q}) \bmod 2\pi} &= \sum_{k=1}^{\infty} Ee^{ip(\Theta_1 + \dots + \Theta_k) \bmod 2\pi} q(1-q)^{k-1} \\ &= \sum_{k=1}^{\infty} \tilde{\phi}_p^k q(1-q)^{k-1}. \end{aligned} \quad (87)$$

After summing up the above geometric series, substituting the form of the wrapped Laplace ch.f. (24), and simplifying, we can write the above ch.f. as follows:

$$\frac{q\tilde{\phi}_p}{1 - (1-q)\tilde{\phi}_p} = \frac{1}{1 + p^2/(q\lambda_q) - ip(1/\kappa_q - \kappa_q)/(q\lambda_q)}. \quad (88)$$

But in view of the relations (53)-(54), we recognize (88) as the ch.f. of the $WL(\lambda, \kappa)$ distribution.

10. Tables

Table 1. Trigonometric moments and related parameters of the wrapped exponential distribution $WE(\lambda)$

Parameter	Definition	Value
Trig. Moments	$\phi_p = \alpha_p + i\beta_p$ $p = 0, \pm 1, \pm 2, \dots$	$\alpha_p = \frac{\lambda^2}{\lambda^2 + p^2}$ $\beta_p = \frac{p\lambda}{\lambda^2 + p^2}$
ρ_p and μ_p^0	$\phi_p = \rho_p e^{i\mu_p^0}, p \geq 0,$	$\rho_p = \frac{ \lambda }{\sqrt{\lambda^2 + p^2}}$ $\mu_p^0 = \begin{cases} \tan^{-1} \frac{p}{\lambda}, & \lambda > 0 \\ 2\pi + \tan^{-1} \frac{p}{\lambda}, & \lambda < 0 \end{cases}$
Resultant length	$\rho = \rho_1$	$\frac{ \lambda }{\sqrt{\lambda^2 + 1}}$
Mean direction	$\mu_0 = \mu_1^0$	$\tan^{-1} \frac{1}{\lambda}, \lambda > 0$ $2\pi + \tan^{-1} \frac{1}{\lambda}, \lambda < 0$
Circular variance	$V_0 = 1 - \rho$	$\frac{\sqrt{1 + \lambda^2} - \lambda }{\sqrt{1 + \lambda^2}}$
Circular standard deviation	$\sigma_0 = \sqrt{-2 \ln(1 - V_0)}$	$\sqrt{\ln(1 + 1/\lambda^2)}$
Central trig. Moments	$\bar{\alpha}_p = \rho_p \cos(\mu_p^0 - p\mu_0)$ $\bar{\beta}_p = \rho_p \sin(\mu_p^0 - p\mu_0)$	$\bar{\alpha}_p = \frac{ \lambda }{\sqrt{\lambda^2 + p^2}} \cos\left(\tan^{-1} \frac{p}{\lambda} - p \tan^{-1} \frac{1}{\lambda}\right)$ $\bar{\beta}_p = \frac{ \lambda }{\sqrt{\lambda^2 + p^2}} \sin\left(\tan^{-1} \frac{p}{\lambda} - p \tan^{-1} \frac{1}{\lambda}\right)$
Skewness	$\gamma_1^0 = \bar{\beta}_2/V_0^{3/2}$	$\frac{-2\lambda}{(1 + \lambda^2)^{1/4}(4 + \lambda^2)(\sqrt{1 + \lambda^2} - \lambda)^{3/2}}$
Kurtosis	$\gamma_2^0 = \frac{\bar{\alpha}_2 - (1 - V_0)^4}{V_0^2}$	$\frac{3\lambda^2}{(1 + \lambda^2)(4 + \lambda^2)(\sqrt{1 + \lambda^2} - \lambda)^2}$
Median	See (44)-(45)	$\frac{1}{\lambda} \ln \frac{2}{1 + e^{-\lambda\pi}} + \begin{cases} 0, & \lambda > 0 \\ \pi, & \lambda < 0 \end{cases}$

Table 2. Trigonometric moments and related parameters of the wrapped Laplace distribution $WL(\lambda, \kappa)$. For the definitions see Table 1. The entries in the last column correspond to the symmetric case with $\kappa = 1$

Parameter(s)	Value(s)	Case $\kappa = 1$
Trig. Moments	$\alpha_p = \frac{\kappa^2 \lambda^2 (p^2 + \lambda^2)}{(\lambda^2 \kappa^2 + p^2)(p^2 \kappa^2 + \lambda^2)}$ $\beta_p = \frac{p\kappa \lambda^3 (1 - \kappa^2)}{(\lambda^2 \kappa^2 + p^2)(p^2 \kappa^2 + \lambda^2)}$	$\alpha_p = \frac{\lambda^2}{p^2 + \lambda^2}$ $\beta_p = 0$
ρ_p and μ_p^0	$\rho_p = \frac{\lambda^2}{\sqrt{\lambda^2 \kappa^2 + p^2} \sqrt{\lambda^2/\kappa^2 + p^2}}, p \geq 0$ $\mu_p^0 = \begin{cases} \tan^{-1} \frac{p}{\lambda\kappa} - \tan^{-1} \frac{p\kappa}{\lambda}, & \kappa \leq 1 \\ 2\pi + \tan^{-1} \frac{p}{\lambda\kappa} - \tan^{-1} \frac{p\kappa}{\lambda}, & \kappa > 1 \end{cases}$	$\rho_p = \frac{\lambda^2}{\lambda^2 + p^2}$ $\mu_p^0 = 0$
Resultant length	$\rho = \frac{\lambda^2}{\sqrt{1 + \lambda^2 \kappa^2} \sqrt{\lambda^2/\kappa^2 + 1}}$	$\rho = \frac{\lambda^2}{\lambda^2 + 1}$
Mean direction	$\mu_0 = \begin{cases} \tan^{-1} \frac{1}{\lambda\kappa} - \tan^{-1} \frac{\kappa}{\lambda}, & \kappa \leq 1 \\ 2\pi + \tan^{-1} \frac{1}{\lambda\kappa} - \tan^{-1} \frac{\kappa}{\lambda}, & \kappa > 1 \end{cases}$	$\mu_0 = 0$
Circular variance	$V_0 = 1 - \frac{\lambda^2}{\sqrt{1 + \lambda^2 \kappa^2} \sqrt{1 + \lambda^2/\kappa^2}}$	$V_0 = \frac{1}{1 + \lambda^2}$
Circular standard deviation	$\sigma_0 = \sqrt{\ln\left(\kappa^2 + \frac{1}{\lambda^2}\right) + \ln\left(\frac{1}{\kappa^2} + \frac{1}{\lambda^2}\right)}$	$\sigma_0 = \sqrt{2 \ln\left(1 + \frac{1}{\lambda^2}\right)}$
Central trig. Moments	$\bar{\alpha}_p = \rho_p \cos(\mu_p^0 - p\mu_0)$ $\bar{\beta}_p = \rho_p \sin(\mu_p^0 - p\mu_0)$	$\bar{\alpha}_p = \frac{\lambda^2}{\lambda^2 + p^2}$ $\bar{\beta}_p = 0$
Skewness	$\gamma_1^0 = \bar{\beta}_2/V_0^{3/2}$, where $\bar{\beta}_2 = \frac{2\lambda^3 \kappa (\lambda^2 \kappa^2 + 3) - 2(\lambda^3/\kappa)(\lambda^2/\kappa^2 + 3)}{(1 + \lambda^2 \kappa^2)(4 + \lambda^2 \kappa^2)(1 + \lambda^2/\kappa^2)(4 + \lambda^2/\kappa^2)}$	$\gamma_1^0 = 0$
Kurtosis	$\gamma_2^0 = \frac{\bar{\alpha}_2 - (1 - V_0)^4}{V_0^2}$, where $\bar{\alpha}_2 = \frac{\lambda^4 (\lambda^2 \kappa^2 + 3)(\lambda^2/\kappa^2 + 3) + 4\lambda^4}{(1 + \lambda^2 \kappa^2)(4 + \lambda^2 \kappa^2)(1 + \lambda^2/\kappa^2)(4 + \lambda^2/\kappa^2)}$	$\gamma_2^0 = (1 + \lambda^2)\lambda^2$ $\times \left(\frac{1}{4 + \lambda^2} - \frac{\lambda^6}{(1 + \lambda^2)^2}\right)$

Median	$\xi_0 = \begin{cases} \xi^*, & \lambda > 0, 0 < \kappa < 1 \\ \xi^* + \pi, & \lambda > 0, \kappa > 1, \end{cases}$ <p>with $\xi^* \in [0, \pi]$ such that</p> $\frac{1}{1 + \kappa^2} \left(\frac{e^{-\lambda \kappa \xi}}{1 + e^{-\lambda \kappa \pi}} + \kappa^2 \frac{e^{\lambda \xi / \kappa}}{1 + e^{\lambda \pi / \kappa}} \right) = \frac{1}{2}$	$\xi_0 = 0$
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